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On the Quasi-Diagonalization and Uncoupling of Gyroscopic Circulatory Multi-Degree-of-Freedom Systems

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A new central result that gives the necessary and sufficient conditions for two n by n skew-symmetric matrices and one symmetric matrix to be simultaneously quasi-diagonalized by a real orthogonal congruence is proved. Based on this result, the decomposition of linear multi-degree-of-freedom dynamical systems with gyroscopic, circulatory, and potential forces is investigated through a real linear coordinate transformation generated by an orthogonal matrix. Several sets of conditions, applicable to real-life structural and mechanical systems arising in aerospace, civil, and mechanical engineering, under which such a coordinate transformation exists are found, thereby allowing these systems to be decomposed into independent, uncoupled subsystems, each with a maximum of two degrees of freedom. The conditions are expressed in terms of the coefficient matrices of the system. A specific form for the circulatory (gyroscopic) matrix is posited, and when the gyroscopic (circulatory) matrix is simple—a situation that commonly appears in real-life applications—it is shown that just a single necessary and sufficient condition is required for the decomposition of the multi-degree-of-freedom system. Numerical examples are provided throughout to demonstrate the analytical results. [DOI: 10.1115/1.4067148]

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1 Introduction

An important class of linear multi-degree-of-freedom (MDOF) dynamical systems is associated with potential (conservative), circulatory and gyroscopic forces and can be described by

$$\tilde{M}\ddot{q} + \tilde{G}\dot{q} + \tilde{N}q + \tilde{K}q = 0 \tag{1}$$

where $\tilde{M}$, $\tilde{G}$, $\tilde{N}$ and $\tilde{K}$ are n by n constant real matrices; the inertia matrix $\tilde{M}$ is symmetric and positive definite ($\tilde{M} = \tilde{M}^T > 0$), $\tilde{G}$ and $\tilde{N}$ are skew-symmetric ($\tilde{G} = -\tilde{G}^T$, $\tilde{N} = -\tilde{N}^T$) and $\tilde{K}$ is symmetric. The n-vector of generalized coordinates is denoted by q, and the dots indicate differentiation with respect to the time t. The terms in $\tilde{G}$, $\tilde{N}$ and $\tilde{K}$ correspond to gyroscopic, circulatory, and potential forces, respectively. The matrices $\tilde{N}$ and $\tilde{K}$ can also be thought of as the skew-symmetric and symmetric additive parts, respectively, of a given stiffness matrix. Since any arbitrary matrix can be uniquely expressed as the sum of a skew-symmetric and a symmetric parts, (1) also describes a multi-degree-of-freedom system, whose stiffness matrix is arbitrary (non-conservative), subjected to a gyroscopic force. Physical systems modeled by (1) are commonly found in aerospace, mechanical, and civil engineering.

Equation (1) represents a set of coupled second-order ordinary-differential equations and can be obtained by the application of Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\partial \Phi}{\partial \dot{q}} = 0 \tag{2}$$

with the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T\tilde{M}\dot{q} + \frac{1}{2}\dot{q}^T\tilde{G}q - \frac{1}{2}q^T\tilde{K}q \tag{3}$$

and the so-called "dissipative" function

$$\Phi(q, \dot{q}) = \dot{q}^T\tilde{N}q \tag{4}$$

Consider a change of coordinates from q to p defined by the real linear transformation

$$q = Pp \Leftrightarrow p = P^{-1}q \tag{5}$$

where P can be any nonsingular real matrix. Noting (3) and (4), this transformation of coordinates causes $\tilde{M}$, $\tilde{K}$, $\tilde{N}$ and $\tilde{G}$ to be

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congruently transformed, i.e.,

$$\tilde{M} \to M = P^T \tilde{M} P, \tilde{K} \to K = P^T \tilde{K} P,$$
$$\tilde{G} \to G = P^T \tilde{G} P, \tilde{N} \to N = P^T \tilde{N} P \tag{6}$$

If $\tilde{N} = 0$ and $\tilde{G} = 0$ (pure potential system or conservative non-gyroscopic system), one can always find the transformation matrix P so that $M = I$ (the identity matrix) and the new potential (stiffness) matrix is diagonal, i.e., in new coordinates p, called normal (principal or modal) coordinates, the system is then decomposed into n independent single-degree-of-freedom subsystems. The procedure for decoupling such systems is well known and is called modal analysis. When $\tilde{N} \neq 0$ and/or $\tilde{G} \neq 0$, the system is not completely decomposable because changes of coordinates (5) that makes M and K diagonal retains N and/or G as skew-symmetric matrices. However, sometimes it might be decomposed into several independent subsystems. Also, it is worth pointing out that the minimum number of degrees of freedom necessary to incorporate circulatory and/or gyroscopic forces is two. Therefore, it is natural to ask whether or not we can decompose system (1) into independent subsystems, each of which has no more than two degrees of freedom, by means of a real change of coordinates (i.e., using a congruence transformation). The intent of this paper is to show that multi-degree of freedom gyroscopic circulatory potential systems can be uncoupled when certain conditions are satisfied.

For a given set of matrices $\tilde{M}, \tilde{K}, \tilde{G},$ and $\tilde{N}$ in (1), our overall aim is to find a real nonsingular matrix P, and the necessary and sufficient conditions under which it exists, such that the matrices $M, K, G,$ and N in (6) are in canonical (simplest) form. The canonical form for M will turn out to be the identity matrix, and for K a diagonal matrix. The canonical forms for G and N will be quasi-diagonal, a term that will be explained in the following section. We will show that when certain necessary and sufficient conditions are satisfied, the multi-degree-of-freedom (MDOF) system described by (1) can be decomposed, through the use of a real coordinate transformation, into independent, uncoupled subsystems each of which has at most two degrees-of-freedom.

Though we have considered the unforced equation of motion (1), all the results related to decomposition and uncoupling in this paper also apply directly to systems that are forced, as will become obvious as we proceed.

In the next section, we formulate an algebraic result, which is basic to our further considerations. In Sec. 3, we develop the necessary and sufficient conditions for the uncoupling of systems under consideration based on our results in Sec. 2. In Sec. 4, we consider a specific but useful form for the circulatory matrix that reduces the number of conditions required for the uncoupling. Section 5 gives the conclusions.

2 Central Theorem

It is well known that for an $n \times n$ real skew-symmetric matrix G there exists an n by n real orthogonal matrix Q such that

$$Q^T G Q = diag(\beta_1 J_2, \ldots, \beta_{n/2} J_2) \text{ for } n \text{ even} \tag{7a}$$

$$= diag(\beta_1 J_2, \ldots, \beta_{(n-1)/2} J_2, 0) \text{ for } n \text{ odd} \tag{7b}$$

where J_2 is the two-dimensional skew-symmetric matrix

$$J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and some of the real numbers β_j may be zero (see, for example [1], p. 65). Furthermore, $J_2^2 = -I_2$, where I_2 is the 2 by 2 identity matrix.

The block-diagonal form of matrix (7), which we shall refer to as *quasi-diagonal*, is the simplest possible (canonical) form of a skew-symmetric matrix with respect to orthogonal similarities, while the canonical form for a real symmetric matrix is, of course, a diagonal matrix consisting of its eigenvalues along the diagonal. We also note that the form (7) is recognized as the real Jordan form for the matrix G and therefore plays a fundamental algebraic role [2].

Lemma 1. *Let $K = K^T$ and $G = -G^T$ be n by n real matrices. The necessary and sufficient conditions that there exists a real orthogonal matrix Q such that*

$$Q^T K Q = \Lambda = diag(\lambda_1, \ldots, \lambda_n) \tag{8}$$

and

$$Q^T G Q = \Gamma = diag(\beta_1 J_2, \ldots, \beta_{n/2} J_2) \text{ for } n \text{ even} \tag{9a}$$

$$= diag(\beta_1 J_2, \ldots, \beta_{(n-1)/2} J_2, 0) \text{ for } n \text{ odd} \tag{9b}$$

where all the λ_j' s and β_j' s are real numbers, are

$$[K, G^2] = 0, \text{or } KG^2 = G^2 K \tag{10}$$

and

$$[K, GKG] = 0, \text{or } (KG)^2 = (GK)^2 \tag{11}$$

where the commutator of any two square matrices A and B is defined as $[A, B] := AB - BA$.

Proof. The proof of this result can be found in Refs. [3] and [4].■

Remark 1. If $Rank(G) = 2m \leq n$ (the rank of a skew-symmetric matrix must be even), then m of the β_j are nonzero. The two-dimensional blocks appearing along the diagonal of matrix (9) can then be ordered, with no loss of generality, in such a way that the first m of them are nonzero, i.e., we can put

$$\Gamma = diag(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}),$$

where the real numbers $\beta_j \neq 0, j = 1, \ldots, m$, and 0_{n-2m} is an $(n - 2m)$ *by* $(n - 2m)$ zero matrix. The nonzero numbers β_j correspond to (complex) conjugate pairs of purely imaginary eigenvalues of G, namely, $\pm\beta_j i$, $i = \sqrt{-1}$, with the zero eigenvalue of G having a multiplicity of $(n - 2m)$.

We next prove a central theorem that plays a key role in our further considerations.

Theorem. *Let $K = K^T$, $G = -G^T$ and $N = -N^T$ be n by n real matrices, and let $Rank(G) = 2m \leq n$. The necessary and sufficient conditions for a real orthogonal matrix Q to exist such that*

$$Q^T K Q = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \tag{12}$$

$$Q^T G Q = \Gamma = \text{diag}(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \tag{13}$$

and

$$Q^T N Q = \mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_{n/2} J_2) \text{ for } n \text{ even} \tag{14a}$$

$$= diag(\nu_1 J_2, \ldots, \nu_{(n-1)/2} J_2, 0) \text{ for } n \text{ odd} \tag{14b}$$

where λ_j, ν_j, and β_j are real numbers with $\beta_j \neq 0, j = 1, \ldots, m,$ are that the following six commutation conditions be met:

$$[G, N] = 0 \tag{15}$$

$$[K, GN] = 0 \tag{16}$$

$$[K, G^2] = 0, \quad [K, GKG] = 0 \tag{17}$$

and

$$[K, N^2] = 0, \quad [K, NKN] = 0 \tag{18}$$

Note the distinction between N and N in (14): N is a real skew-symmetric matrix, while N is a quasi-diagonal, skew-symmetric matrix whose structure is given in (14). This notation will be used throughout the paper.

To prove this theorem, we additionally need the following two lemmas.

LEMMA 2. *The product of two skew-symmetric matrices G and N is symmetric if and only if* $[G, N] = 0$.

Proof. Assume that $[G, N] = 0$, then $GN = NG = (GN)^T$. The last equality follows because G and N are both skew-symmetric. Hence, GN is symmetric. On the other hand, if GN is symmetric then $GN = (GN)^T = NG$; the last equality again follows because G and N are skew-symmetric. Hence $GN - NG = [G, N] = 0$. ■

LEMMA 3. *If conditions (15)–(17) are satisfied, then the symmetric matrices*

$$K, GN, G^2, GKG \tag{19}$$

commute pairwise.

Proof. According to Lemma 2, the matrix GN is symmetric; the other three matrices of the family shown in (19) are symmetric, since the matrix G is skew-symmetric, and K is symmetric. There is a total of $C_2^4 = 6$ conditions for pairwise commutation of the matrices in (19). Three of them appear in (16) and (17); the remaining three are shown to follow from (15)–(17) in the Appendix. ■

Remark 2. When Λ is diagonal as in (12), and Γ and N are quasi-diagonal as in (13) and (14), then the matrices $\Gamma\mathrm{N}$, Γ^2, N^2, $\Gamma\Lambda\Gamma$ and $\mathrm{N}\Lambda\mathrm{N}$ are diagonal. They are:

$$\Gamma\mathrm{N} = \mathrm{N}\Gamma = -diag(\beta_1\nu_1 I_2, \ldots, \beta_m\nu_m I_2, 0_{n-2m}) \tag{20}$$

$$\Gamma^2 = -diag(\beta_1^2 I_2, \ldots, \beta_m^2 I_2, 0_{n-2m}), \beta_j \neq 0, j = 1, \ldots, m \tag{21}$$

$$\mathrm{N}^2 = -diag(\nu_1^2 I_2, \ldots, \nu_{n/2}^2 I_2,) \; for \; n \; even \tag{22a}$$

$$= -\, diag(\nu_1^2 I_2, \ldots, \nu_{(n-1)/2}^2 I_2, 0) \; for \; n \; odd \tag{22b}$$

$$\Gamma\Lambda\Gamma = -diag(\beta_1^2\lambda_2, \beta_1^2\lambda_1, \ldots, \beta_m^2\lambda_{2m}, \beta_m^2\lambda_{2m-1}, 0_{n-2m}) \tag{23}$$

and

$$\mathrm{N}\Lambda\mathrm{N} = -diag(\nu_1^2\lambda_2, \nu_1^2\lambda_1, \ldots, \nu_{n/2}^2\lambda_n, \nu_{n/2}^2\lambda_{n-1}) \; for \; n \; even \tag{24a}$$

$$= -\, diag(\nu_1^2\lambda_2, \nu_1^2\lambda_1, \ldots, \nu_{(n-1)/2}^2\lambda_{(n-1)}, \nu_{(n-1)/2}^2\lambda_{(n-2)}, 0) \; for \; n \; odd \tag{24b}$$

Proof. Let us prove (20). Since Γ and N are block-diagonal matrices, the product ΓN is obtained by multiplying the corresponding diagonal blocks of Γ and N. Premultiplying the j^{th} diagonal block of N (Γ) by the j^{th} diagonal block of Γ (N) we get the j^{th} diagonal block of the product $\Gamma\mathrm{N}$ ($\mathrm{N}\Gamma$), i.e., for $j = 1, \ldots, m$

$$[\beta_j J_2][v_j J_2] = \beta_j v_j J_2^2 = v_j \beta_j J_2^2 = [v_j J_2][\beta_j J_2] = -\beta_j v_j I_2$$

The third equality shows that the jth diagonal block of $\Gamma\mathrm{N}$ is the same as the jth diagonal block of $\mathrm{N}\Gamma$, each being $-\beta_j v_j I_2$, and therefore

$$\Gamma\mathrm{N} = \mathrm{N}\Gamma = -diag(\beta_1\nu_1 I_2, \ldots, \beta_m\nu_m I_2, 0_{n-2m})$$

where some of the products $\beta_j v_j$ could be zero because some of the v_j could be zero.

When n is even, Λ can also be considered a block diagonal matrix with each block being a 2 by 2 diagonal matrix and by multiplication of the corresponding blocks the products shown in (21)–(24) can be similarly obtained. When n is odd, the last block of Λ is one-dimensional (scalar) and the multiplication yields the result shown. ■

Proof of the Theorem. Necessity. Let the orthogonal matrix Q be such that $Q^TKQ = \Lambda$, $Q^TGQ = \Gamma$ and $Q^TNQ = \mathrm{N}$ with Λ, Γ and N as in (12), (13) and (14), respectively. Then $[G, N] = Q[\Gamma, \mathrm{N}]Q^T = 0$ because Γ and N commute (see (20)). Also, we have $[K, GN] = Q[\Lambda, \Gamma\mathrm{N}]Q^T = 0$, $[K, G^2] = Q[\Lambda, \Gamma^2]Q^T = 0$, $[K, GKG] = Q[\Lambda, \Gamma\Lambda\Gamma]Q^T = 0$, $[K, N^2] = Q[\Lambda, \mathrm{N}^2]Q^T = 0$ and $[K, NKN] = Q[\Lambda, \mathrm{N}\Lambda\mathrm{N}]Q^T = 0$ since, according to *Remark 2*, all the matrices $\Gamma\mathrm{N}$, Γ^2, N^2, $\Gamma\Lambda\Gamma$ and $\mathrm{N}\Lambda\mathrm{N}$ are diagonal, and therefore commute with the diagonal matrix Λ, making all their commutators equal to zero.

Sufficiency. Suppose that conditions (15)–(18) are satisfied. Then, because of Lemma 3, the symmetric matrices in (19) commute pairwise and, according to a well-known result (see, for example, Ref. [2]), they have n common linearly independent eigenvectors. Let $\sigma(G) = (\pm i\beta_1, \ldots, \pm i\beta_m, 0, \ldots, 0)$, $\beta_j \neq 0$, $j = 1, 2, \ldots, m$, be the spectrum (denoted by $\sigma(\square)$) of the skew-symmetric matrix G. Then $\sigma(G^2) = (-\beta_1^2, -\beta_1^2, \ldots, -\beta_m^2, -\beta_m^2, 0, \ldots, 0)$. With no loss of generality, let q_1 be a (real) unit eigenvector such that

$$G^2q_1 = -\beta_1^2 q_1, \beta_1 \neq 0$$

$$Kq_1 = \lambda_1 q_1$$

$$GKGq_1 = \mu_1 q_1$$

and

$$GNq_1 = \alpha_1 q_1$$

where, λ_1, μ_1, and α_1 are real numbers, which could be zero. Premultiplying each of the last two equations by G gives $G^2KGq_1 = \mu_1 Gq_1$ and $G^2Nq_1 = \alpha_1 Gq_1$. Since $G^2K = KG^2$ (because G^2 and K commute) and $G^2N = NG^2$ (because G and N commute), we then get $KGG^2q_1 = \mu_1 Gq_1$ and $NG^2q_1 = \alpha_1 Gq_1$. Furthermore, because $G^2q_1 = -\beta_1^2 q_1$, these two relations become

$$K(Gq_1) = -\mu_1\beta_1^{-2}(Gq_1) \tag{25}$$

and

$$Nq_1 = -\alpha_1\beta_1^{-2}Gq_1 \tag{26}$$

Since $\|Gq_1\| = \sqrt{q_1^TG^TGq_1} = \sqrt{-q_1^TG^2q_1} = \sqrt{\beta_1^2 q_1^Tq_1} = \beta_1 \neq 0$, it follows from (25) that the vector $q_2 := -\beta_1^{-1}Gq_1$ is a unit eigenvector of the matrix K corresponding to the eigenvalue $\lambda_2 := -\mu_1\beta_1^{-2}$. Furthermore, because G is skew-symmetric, $q_1^Tq_2 = -\beta_1^{-1}q_1^TGq_1 = 0$, i. e., the unit vectors q_1 and q_2 are orthogonal. On the other hand, we see from (26) that $Nq_1 = -\nu_1 q_2$ with $\nu_1 := -\beta_1^{-1}\alpha_1$. Also, $Nq_2 = -\beta_1^{-1}NGq_1 = -\beta_1^{-1}\alpha_1 q_1 = \nu_1 q_1$ since G and N commute. Now using q_1 and q_2 as the first and second columns we form an orthogonal matrix $Q_1 = \begin{bmatrix} q_1 & q_2 & q_3 & \ldots & q_n \end{bmatrix}$, whose remaining columns can be chosen arbitrarily provided $Q_1^TQ_1 = I_n$.

We see that for $k = 1, 2, \ldots, n$, noting the orthogonality of the columns of Q_1, the elements of the first and second rows (columns) of the symmetric matrix $Q_1^TKQ_1$ are given by

$$\begin{aligned} q_1^TKq_k &= q_k^TKq_1 = \lambda_1 q_k^Tq_1 = \lambda_1\delta_{1k} \text{ and} \\ q_2^TKq_k &= q_k^TKq_2 = \lambda_2 q_k^Tq_2 = \lambda_2\delta_{2k} \end{aligned} \tag{27}$$

where δ_{jk} denotes the Knonecker delta. We note that $Gq_1 = -\beta_1 q_2$. Multiplying both sides of this equation by G, we infer that $Gq_2 = -\beta_1^{-1}G^2q_1 = \beta_1 q_1$.

Hence, for $k = 1, 2, \ldots, n$, the elements of the first and second rows (columns) of the skew-symmetric matrix $Q_1^TGQ_1$ are then given, respectively, by the relations

$$\begin{aligned} q_1^TGq_k &= -q_k^TGq_1 = \beta_1 q_k^Tq_2 = \beta_1\delta_{2k} \text{ and} \\ q_2^TGq_k &= -q_k^TGq_2 = -\beta_1 q_k^Tq_1 = -\beta_1\delta_{1k} \end{aligned} \tag{28}$$

Similarly, for $k=1,2,\ldots,n$, noting that $Nq_1=-\nu_1 q_2$ and $Nq_2=\nu_1 q_1$, the first two rows (columns) of the skew-symmetric matrix $Q_1^T N Q_1$ are given by

$$q_1^T N q_k = -q_k^T N q_1 = \nu_1 q_k^T q_2 = \nu_1 \delta_{2k} \text{ and}$$
$$q_2^T N q_k = -q_k^T N q_2 = -\nu_1 q_k^T q_1 = -\nu_1 \delta_{1k} \tag{29}$$

From (27)–(29), the structures of the three matrices $Q_1^T K Q_1$, $Q_1^T G Q_1$ and $Q_1^T N Q_1$ are thus found to be as follows:

$$Q_1^T K Q_1 = \left[\begin{array}{cc|c} \lambda_1 & 0 & \\ 0 & \lambda_2 & 0 \\ \hline \multicolumn{2}{c|}{0} & K_{n-2} \end{array}\right]$$

$$Q_1^T G Q_1 = \left[\begin{array}{cc|c} 0 & \beta_1 & \\ -\beta_1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & G_{n-2} \end{array}\right]$$

and

$$Q_1^T N Q_1 = \left[\begin{array}{cc|c} 0 & \nu_1 & \\ -\nu_1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & N_{n-2} \end{array}\right].$$

Since the $(n-2)$-dimensional matrices K_{n-2}, G_{n-2} and N_{n-2} satisfy the same conditions as K, G and N this procedure continues in the same manner, and after m steps we conclude that there exists an orthogonal matrix $\hat{Q}$ such that

$$\hat{Q}^T G \hat{Q} = diag\left(\beta_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0_{n-2m}\right)$$

$$\hat{Q}^T K \hat{Q} = diag(\lambda_1, \ldots, \lambda_{2m}, K_{n-2m})$$

and

$$\hat{Q}^T N \hat{Q} = diag\left(\nu_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \nu_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, N_{n-2m}\right)$$

where 0_{n-2m} is $(n-2m)$-dimensional zero matrix, while K_{n-2m} and N_{n-2m} are $(n-2m)$-dimensional symmetric and skew-symmetric matrices, respectively. From (18), the matrices K_{n-2m} and N_{n-2m} satisfy conditions

$$[K_{n-2m}, N_{n-2m}^2] = 0 \text{ and } [K_{n-2m}, N_{n-2m}K_{n-2m}N_{n-2m}] = 0$$

and, according to Lemma 1, there exists a real orthogonal $(n-2m)$-dimensional matrix $\bar{Q}_{n-2m}$ that reduces the matrices K_{n-2m} and N_{n-2m} simultaneously to diagonal and quasi-diagonal forms, respectively, i. e.,

$$\bar{Q}_{n-2m}^T K_{n-2m} \bar{Q}_{n-2m} = diag(\lambda_{2m+1}, \ldots, \lambda_n)$$

and

$$\bar{Q}_{n-2m}^T N_{n-2m} \bar{Q}_{n-2m} = diag(\nu_{m+1}J_2, \ldots, \nu_{n/2}J_2) \text{ for } n \text{ even}$$
$$= diag(\nu_{m+1}J_2, \ldots, \nu_{(n-1)/2}J_2, 0) \text{ for } n \text{ odd}$$

Thus, the orthogonal matrix

$$Q = \hat{Q}\begin{bmatrix} I_{2m} & 0 \\ 0 & \bar{Q}_{n-2m} \end{bmatrix}$$

simultaneously reduces K, G and N to the forms given in (12)–(14). ∎

When the forms in (12)–(14) are obtained we shall refer to this as the *simultaneous (orthogonal) quasi-diagonalization of the matrices* G, N, and K by the (real orthogonal) matrix Q.

Remark 3. It is clear from the proof of the Theorem that the roles of G and N can be interchanged.

Remark 4. It can be verified by direct computation that the six conditions in (15)–(18) are satisfied when $n=2$.

Remark 5. It follows from the proof of the theorem that only the four conditions given in (15)–(17) are necessary and sufficient for a real orthogonal matrix $\hat{Q}$ to exist that simultaneously reduces G, K and N to the block-diagonal forms

$$\hat{Q}^T G \hat{Q} = diag\left(\beta_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0_{n-2m}\right),$$

$$\hat{Q}^T K \hat{Q} = diag(\lambda_1, \ldots, \lambda_{2m}, K_{n-2m})$$

and

$$\hat{Q}^T N \hat{Q} = diag\left(\nu_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \nu_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, N_{n-2m}\right)$$

where K_{n-2m} and N_{n-2m} are $(n-2m)$-dimensional symmetric and skew-symmetric matrices, respectively. The additional two conditions given in (18) ensure the simultaneous orthogonal diagonalization of $K(K_{n-2m})$ and quasi-diagonalization of $N(N_{n-2m})$. We show in this Remark that when $2m \geq n-2$, only the four conditions (15)–(17) are necessary and sufficient for the simultaneous orthogonal quasi-diagonalization of G, N, and K.

We begin by noting that when G has full rank ($n=2m$), the three matrices 0_{n-2m}, K_{n-2m}, and N_{n-2m} disappear, and so (15)–(17) give the necessary and sufficient conditions for the simultaneous orthogonal quasi-diagonalization of G, N, and K. When $2m=n-2$, in the above block-diagonal forms the submatrices K_{n-2m} and N_{n-2m} are two-dimensional, and consequently, by Remark 4, conditions (18) are automatically satisfied. In the case when $2m=n-1$ (then n is odd) these three submatrices are one-dimensional (scalars) and they therefore always commute. Therefore, when $2m \geq n-2$, the four conditions in (15)–(17) are necessary and sufficient for the simultaneous orthogonal quasi-diagonalization of the matrices G, N, and K.

A direct consequence of our Theorem is the following assertion. It is a counterpart of the well-known result related to real symmetric matrices which states that two such matrices can be simultaneously diagonalized by *a real orthogonal transformation* if and only if they commute in multiplication [1].

Corollary 1. *Let $G=-G^T$ and $N=-N^T$ be n by n real matrices, and let Rank(G) $=2m \leq n$. The necessary and sufficient condition that there exists a real orthogonal matrix Q such that*

$$Q^T G Q = \Gamma = diag(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \tag{30}$$

and

$$Q^T N Q = \mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_{n/2} J_2) \text{ for } n \text{ even} \tag{31a}$$

$$= diag(\nu_1 J_2, \ldots, \nu_{(n-1)/2} J_2, 0) \text{ for } n \text{ odd} \tag{31b}$$

is that the skew-symmetric matrices G and N commute in multiplication, i.e., $[G, N]=0$.

Proof. Application of the Theorem with $K=0$ gives the result. ∎

We next give three lemmas that will be used when we consider the application of our Theorem in the following Section to real-world structural and mechanical systems.

Lemma 4. *Let $K=K^T$ and $G=-G^T$ be n by n real matrices. If all the nonzero eigenvalues of the skew-symmetric matrix G are distinct, then the condition*

$$[K, G^2] = 0$$

implies the condition

$$[K, GKG] = 0.$$

Proof. See [4]. ∎

LEMMA 5. *Let $K = K^T$, $G = -G^T$ and $N = -N^T$ be n by n real matrices. If all the eigenvalues of the symmetric matrix K are distinct, then the condition*

$$[K, GN] = 0$$

implies the condition

$$[G, N] = 0.$$

Proof. Since K is symmetric there exists a real orthogonal matrix Q such that $K = Q\Lambda Q^T$ where $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. Then from $[K, GN] = 0$ we get $\Lambda Q^T GNQ = Q^T GNQ\Lambda$. Upon expansion, this condition is equivalent to

$$\lambda_r b_{rs} = \lambda_s b_{rs}, r, s = 1, \ldots, n$$

where b_{rs} is the rs^{th} element of the matrix $Q^T GNQ$. Since all λ_r are distinct, the above equations imply that $b_{rs} = 0$ when $r \neq s$. That means that $Q^T GNQ$ is a diagonal matrix and therefore GN is a symmetric matrix. Since G and N are skew-symmetric matrices and their product is symmetric, $[G, N] = 0$ by Lemma 2. ∎

LEMMA 6. *Let $K = K^T$, $G = -G^T$ and $N = -N^T$ be n by n real matrices, and let $Rank(G) = 2m \leq n$. If all the nonzero eigenvalues of the skew-symmetric matrix G are distinct, then the two conditions*

$$[G, N] = 0 \text{ and } [K, G^2] = 0$$

imply the condition

$$[K, GN] = 0$$

Proof. Since G and N commute, according to Corollary 1, there exists a real orthogonal matrix Q such that

$$G = Q\begin{bmatrix} \hat{\Gamma} & 0 \\ 0 & 0_{n-2m} \end{bmatrix}Q^T, \hat{\Gamma} = diag(\beta_1 J_2, \ldots, \beta_m J_2)$$

and $N = Q\mathrm{N}Q^T$ where N is as in (31). Also, we can write

$$K = Q\begin{bmatrix} \hat{K} & \tilde{K} \\ \tilde{K}^T & \bar{K} \end{bmatrix}Q^T$$

where $\hat{K}$ and $\bar{K}$ are $2m$- and $(n-2m)$-dimensional symmetric matrices, respectively, and $\tilde{K}$ is $2m$ by $(n-2m)$ matrix. Then, the condition $[K, G^2] = 0$ yields $\tilde{K} = 0$ because $\hat{\Gamma}$ is nonsingular, and

$$\hat{K}\hat{\Gamma}^2 = \hat{\Gamma}^2\hat{K}$$

Next, after partitioning the symmetric matrix $\hat{K}$, as $\hat{K} = [\hat{K}_{rs}]_{r,s=1}^m$ with two-dimensional sub-matrices $\hat{K}_{rs}$, the above condition becomes

$$\beta_r^2\hat{K}_{rs} = \beta_s^2\hat{K}_{rs}, r, s = 1, \ldots, m$$

which yield $\hat{K}_{rs} = 0$ when $r \neq s$ because, in view of the assumption, the set of numbers $\{\beta_1, \ldots, \beta_m\}$ are distinct. Thus, the matrix K that satisfies the condition $[K, G^2] = 0$ must be of the form

$$K = Q\begin{bmatrix} diag(\hat{K}_{rr})_{r=1}^m & 0 \\ 0 & \bar{K} \end{bmatrix}Q^T$$

where $\hat{K}_{rr}$, $r = 1, \ldots, m$, are two by two symmetric matrices and $\bar{K}$ is an $(n-2m)$-dimensional symmetric matrix. But from (20), we know that $\Gamma\mathrm{N} = -diag(\beta_1\nu_1 I_2, \ldots, \beta_m\nu_m I_2, 0_{n-2m})$ so that

$$\begin{aligned}[K, GN] &= Q[diag(\hat{K}, \bar{K}), \Gamma\mathrm{N}]Q^T \\ &= -Qdiag([\hat{K}_{11}, \beta_1\nu_1 I_2], \ldots, [\hat{K}_{mm}, \beta_m\nu_m I_2], 0_{n-2m})Q^T \\ &= -Qdiag(\beta_1\nu_1[\hat{K}_{11}, I_2], \ldots, \beta_m\nu_m[\hat{K}_{mm}, I_2], 0_{n-2m})Q^T = 0\end{aligned}$$

since $[\hat{K}_{rr}, I_2] = 0$, $r = 1, \ldots, m$. ∎

3 Uncoupling of Gyroscopic Circulatory Systems

We begin with the observation that the change of coordinates $q = \tilde{M}^{-1/2}x$, where $\tilde{M}^{-1/2}$ denotes the inverse of the unique positive definite square root of $\tilde{M}$, transforms (1) to the simpler form

$$\ddot{x} + G\dot{x} + Nx + Kx = 0 \tag{32}$$

where

$$G = -G^T = \tilde{M}^{-1/2}\tilde{G}\tilde{M}^{-1/2} \tag{33}$$

$$N = -N^T = \tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2} \tag{34}$$

and

$$K = K^T = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2} \tag{35}$$

The systems described in (1) and (32) are equivalent and we will deal mainly with (32) in what follows. We shall refer to the matrices K, G, and N as the stiffness (potential) matrix, the gyroscopic matrix, and the circulatory matrix, respectively.

The system described by (1) can be uncoupled into (real) independent subsystems of at most two degrees-of-freedom using a real change of coordinates if and only if it can be uncoupled for the system described by (32) by *a real orthogonal transformation*. On the other hand, observe that in principal coordinates a two degree-of-freedom system under consideration has the canonical (simplest) form

$$\ddot{p} + \beta J_2\dot{p} + \nu J_2 p + \Lambda p = 0 \tag{36}$$

where p is a two-dimensional real vector of principal coordinates, $\Lambda = diag(\lambda_1, \lambda_2)$ and λ_1, λ_2, ν and β are real numbers.

Suppose now that $Rank(\tilde{G}) = 2m \leq n$ and that a transformation (5) decomposes the system (1) into independent subsystems, each of which has no more than two degrees of freedom. In this case, in view of the above observation, we can assume that the transformed system has the following form

$$\ddot{p} + \Gamma\dot{p} + \mathrm{N}p + \Lambda p = 0 \tag{37}$$

with

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{38}$$

$$\Gamma = diag(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \tag{39}$$

and

$$\mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_{n/2} J_2) \text{ for } n \text{ even} \tag{40a}$$

$$= diag(\nu_1 J_2, \ldots, \nu_{(n-1)/2} J_2, 0) \text{ for } n \text{ odd} \tag{40b}$$

where λ_j, ν_j and β_j are real numbers with $\beta_j \neq 0, j = 1, \ldots, m$.

We note that (37) describes a set of *independent*, *uncoupled* (real) subsystems of *at most two* degrees of freedom.

Result 1. Let $K = K^T$, $N = -N^T$, $G = -G^T$ and $Rank(G) = 2m \leq n$. The necessary and sufficient conditions for (32) to be transformed to (37) with Λ, Γ and N as in (38)–(40) using a real orthogonal transformation are

$$[G, N] = 0, [K, GN] = 0 \tag{41}$$

$$[K, G^2] = 0, [K, GKG] = 0 \tag{42}$$

and

$$[K, N^2] = 0, [K, NKN] = 0 \tag{43}$$

Proof. Using an *orthogonal* transformation $x = Qp$ in (32) and multiplying from the left by Q^T gives

$$\ddot{p} + Q^T GQ\dot{p} + Q^T NQp + Q^T KQp = 0$$

The theorem states that an orthogonal matrix Q exists, such that

$$Q^T KQ = \Lambda, Q^T GQ = \Gamma, Q^T NQ = \mathrm{N}$$

where Λ, Γ and N are as in (38)–(40), if and only if conditions (41)–(43) are satisfied. ∎

Note that the uncoupling conditions (41)–(43) trivially hold (disappear) in the following three cases: $G = N = 0$, $K = G = 0$ and $K = N = 0$. In the first case, as well known and previously mentioned in the Introduction, the system can be transformed to the completely uncoupled form $\ddot{p} + \Lambda p = 0$. In the second and third cases the system can be reduced to the forms $\ddot{p} + \mathrm{N}p = 0$ and $\ddot{p} + \Gamma\dot{p} = 0$, respectively. If $N = 0$, Result 1 gives necessary and sufficient conditions for the quasi-diagonalization of conservative gyroscopic systems obtained earlier in Ref. [4].

Remark 6. The roles of G and N can be interchanged in Result 1, as well as in other results of this section. Thus, when $Rank(N) = 2m \leq n$ then the conditions in (41)–(43), in which the symbols G and N are interchanged, are necessary and sufficient for (32) to be transformed using a real orthogonal transformation to (37) with

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$\Gamma = diag(\beta_1 J_2, \ldots, \beta_{n/2} J_2) \textit{ for n even}$$

$$= diag(\beta_1 J_2, \ldots, \beta_{(n-1)/2} J_2, 0) \textit{ for n odd}$$

and

$$\mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_m J_2, 0_{n-2m})$$

where β_j and ν_j are real numbers with $\nu_j \neq 0$, $j = 1, \ldots, m$.

Remark 7. Conditions in (41)–(43) are equivalent to the symmetry of the following set of matrices

$$GN, KGN, KG^2, (KG)^2, KN^2, (KN)^2$$

as well as to the symmetry of GN and the pairwise commutation of the matrices

$$K, GN, G^2, N^2, GNG, NKN$$

Remark 8. Consider the case when one of the two skew-symmetric matrices G or N is nonsingular. For definiteness, let us say that N is nonsingular. Then from Remarks 5 and 6, the four conditions

$$[G, N] = 0, [K, GN] = 0, [K, N^2] = 0, [K, NKN] = 0$$

are necessary and sufficient for system (32) to be transformed by a real orthogonal change of coordinates to (37) with the matrices

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$
$$\Gamma = diag(\beta_1 J_2, \ldots, \beta_m J_2), \text{ and}$$
$$\mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_m J_2)$$

where β_j and ν_j are real numbers with $\nu_j \neq 0$, $j = 1, \ldots, m$. System (32) then uncouples into $m = n/2$ independent two-degree-of-freedom subsystems each of which, in general, is a gyroscopic circulatory system.

On the other hand, if G is nonsingular conditions (41) and (42) are necessary and sufficient for system (32) to be transformed by a real orthogonal transformation to (37). The matrices Λ, Γ, and N have the same structure as above except that now β_j and ν_j are real numbers with $\beta_j \neq 0$, $j = 1, \ldots, m$. This again leads to $m = n/2$ independent two-degree-of-freedom subsystems, as before.

When n is odd then the uncoupled system has at least one single degree of freedom potential subsystem.

Remark 9. The above result regarding uncoupling of the MDOF system can be directly applied to gyroscopic circulatory systems that are forced. Namely, the necessary and sufficient conditions for the equation

$$\ddot{x} + G\dot{x} + Nx + Kx = f(t),$$

where $K = K^T$, $N = -N^T$, $G = -G^T$, $Rank(G) = 2m \leq n$ and the external force n-vector $f(t) = [f_1(t), f_2(t), \ldots, f_n(t)]^T$, to be transformed to

$$\ddot{p} + \Gamma\dot{p} + \mathrm{N}p + \Lambda p = g(t) = :Q^T f(t)$$

with Λ, Γ and N as in (38)–(40) using a real orthogonal transformation $x = Qp$ are conditions (41)–(43).

Let us illustrate Result 1 and Remark 9 with the following two examples.

Example 1. Consider a four-degree-of-freedom system subjected to the external force n-vector $f(t)$, as described above, in which

$$K = \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.25 & 0.75 \\ 0 & 0 & 0.75 & 1.25 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \\ -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{bmatrix},$$
$$N = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{44}$$

To check if the six commutation conditions (41)–(43) are satisfied, we use Remark 7. We obtain:

$$GN = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} = NG, \text{ i.e., } [G, N] = 0;$$

$$KGN = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = GNK, \text{ i.e., } [K, GN] = 0;$$

$$KG^2 = \begin{bmatrix} -12 & 12 & 0 & 0 \\ 12 & -12 & 0 & 0 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & -4 \end{bmatrix} = G^2K, \text{ i.e., } [K,\, G^2] = 0;$$

$$(KG)^2 = \begin{bmatrix} -6 & 6 & 0 & 0 \\ 6 & -6 & 0 & 0 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 6 & -6 \end{bmatrix} = (GK)^2, \text{ i.e., } [K,\, GKG] = 0;$$

$$KN^2 = \begin{bmatrix} -1 & 0.5 & 0 & 0 \\ 0.5 & -1 & 0 & 0 \\ 0 & 0 & -1.25 & -0.75 \\ 0 & 0 & -0.75 & -1.25 \end{bmatrix} = N^2K, \text{ i.e., }$$

$[K,\, N^2] = 0$; and,

$$(KN)^2 = \begin{bmatrix} -0.875 & -0.125 & 0 & 0 \\ -0.125 & -0.875 & 0 & 0 \\ 0 & 0 & -0.875 & -0.125 \\ 0 & 0 & -0.125 & -0.875 \end{bmatrix} = (NK)^2, \text{ i.e., } [K,\, NKN] = 0$$

Thus, conditions (41)–(43) of Result 1 are satisfied, and taking into account that $\det N \neq 0$, the system in this example can be transformed by a real orthogonal transformation into two independent two-dimensional subsystems. Indeed, one easily verifies that the orthogonal coordinate transformation $x = Qp$, where

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

decomposes the system into two independent, uncoupled two-degree-of-freedom subsystems described by

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 1.5p_1 \\ 0.5p_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} f_1(t) - f_2(t) \\ f_3(t) - f_4(t) \end{bmatrix}$$

and

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_3 \\ p_4 \end{bmatrix} + \begin{bmatrix} 0.5p_3 \\ 2p_4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} f_1(t) + f_2(t) \\ f_3(t) + f_4(t) \end{bmatrix}.$$

Example 2. We consider a nine-degree-of-freedom system that is subjected to an external force n-vector $f(t)$ and that is described by the matrices (for brevity, the numbers shown below are correct to 4 decimal places)

$$K = \begin{bmatrix} 7.1032 & -0.1111 & -0.1111 & -0.3968 & -0.5635 & -0.5635 & -1.0635 & -1.3968 & -1.8968 \\ -0.1111 & 1.8889 & -0.1111 & -0.1111 & -0.1111 & -0.1111 & -0.1111 & -0.1111 & -0.1111 \\ -0.1111 & -0.1111 & 1.8889 & -0.1111 & -0.1111 & -0.1111 & -0.1111 & -0.1111 & -0.1111 \\ -0.3968 & -0.1111 & -0.1111 & 3.6032 & -0.3968 & -0.3968 & -0.3968 & -0.3968 & -0.3968 \\ -0.5635 & -0.1111 & -0.1111 & -0.3968 & 4.4365 & -0.5635 & -0.5635 & -0.5635 & -0.5635 \\ -0.5635 & -0.1111 & -0.1111 & -0.3968 & -0.5635 & 4.4365 & -0.5635 & -0.5635 & -0.5635 \\ -1.0635 & -0.1111 & -0.1111 & -0.3968 & -0.5635 & -0.5635 & 5.9365 & -1.0635 & -1.0635 \\ -1.3968 & -0.1111 & -0.1111 & -0.3968 & -0.5635 & -0.5635 & -1.0635 & 6.6032 & -1.3968 \\ -1.8968 & -0.1111 & -0.1111 & -0.3968 & -0.5635 & -0.5635 & -1.0635 & -1.3968 & 7.1032 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & -0.7071 & -0.2887 & 0.2887 & 0 & 0 & 0 & 0 & 0 \\ 0.7071 & 0.0000 & 0.7071 & 0.7071 & 0.7071 & 0.7071 & 0.7071 & 0.7071 & 0.7071 \\ 0.2887 & -0.7071 & 0 & -1.7321 & 0.2887 & 0.2887 & 0.2887 & 0.2887 & 0.2887 \\ -0.2887 & -0.7071 & 1.7321 & 0 & -0.2887 & -0.2887 & -0.2887 & -0.2887 & -0.2887 \\ 0 & -0.7071 & -0.2887 & 0.2887 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.7071 & -0.2887 & 0.2887 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.7071 & -0.2887 & 0.2887 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.7071 & -0.2887 & 0.2887 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.7071 & -0.2887 & 0.2887 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.6124 & -0.6124 & -1.0607 & 1.0607 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.6124 & 0 & 0 & 0 & 0 & 2.4495 & 0 & -0.6124 & -0.6124 \\ 0.6124 & 0 & 0 & 0 & -2.4495 & 0 & 0.6124 & 0.6124 & 0.6124 \\ 1.0607 & 0 & 0 & 0 & 0.6124 & -0.6124 & 0 & -2.1213 & 1.0607 \\ -1.0607 & 0 & 0 & 0 & 0.6124 & -0.6124 & 2.1213 & 0 & -1.0607 \\ 0 & 0 & 0 & 0 & 0.6124 & -0.6124 & -1.0607 & 1.0607 & 0 \end{bmatrix}$$

and a computation shows that conditions (41)–(43) are satisfied. The eigenvalues of K are $\{1, 2, 2, 4, 5, 5, 7, 8, 9\}$; they are not distinct, since there are two eigenvalues with multiplicity 2. Also, the eigenvalues of G are $\{2i, -2i, 2i, -2i, 0, 0, 0, 0, 0\}$, and the eigenvalues of N are $\{3i, -3i, 3i, -3i, 0, 0, 0, 0, 0\}$; the non-zero eigenvalues of both these matrices are therefore not distinct. The rank of both G and N is 4.

The orthogonal matrix

$$Q=\begin{bmatrix} a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1\\ a_9 & -8a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ a_9 & a_8 & -7a_7 & 0 & 0 & 0 & 0 & 0 & 0\\ a_9 & a_8 & a_7 & -6a_6 & 0 & 0 & 0 & 0 & 0\\ a_9 & a_8 & a_7 & a_6 & -5a_5 & 0 & 0 & 0 & 0\\ a_9 & a_8 & a_7 & a_6 & a_5 & -4a_4 & 0 & 0 & 0\\ a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & -3a_3 & 0 & 0\\ a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & -2a_2 & 0\\ a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & -a_1 \end{bmatrix}$$

where $a_9 = 1/3$ and $a_j = 1/\sqrt{(j+j^2)}$, $j = 1, 2, \ldots, 8$, simultaneously quasi-diagonalized K, G, and N, and gives

$$\Lambda = diag(1, 2, 2, 4, 5, 5, 7, 8, 9),$$
$$\Gamma = diag(2J_2, -2J_2, 0, 0, 0, 0, 0)$$

and

$$\mathrm{N} = diag(0, 0, 0, 0, 3J_2, -3J_2, 0)$$

so that the forced system with these matrices K, G, and N uncouples into the following four independent two-degree-of-freedom subsystems, and one independent single-degree-of-freedom subsystem:

$$\begin{bmatrix}\ddot p_1\\ \ddot p_2\end{bmatrix}+\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}\dot p_1\\ \dot p_2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}=\begin{bmatrix}g_1(t)\\ g_2(t)\end{bmatrix}$$

$$\begin{bmatrix}\ddot p_3\\ \ddot p_4\end{bmatrix}+\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}\begin{bmatrix}\dot p_3\\ \dot p_4\end{bmatrix}+\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}p_3\\ p_4\end{bmatrix}=\begin{bmatrix}g_3(t)\\ g_4(t)\end{bmatrix}$$

$$\begin{bmatrix}\ddot p_5\\ \ddot p_6\end{bmatrix}+\begin{bmatrix}5 & 3\\ -3 & 5\end{bmatrix}\begin{bmatrix}p_5\\ p_6\end{bmatrix}=\begin{bmatrix}g_5(t)\\ g_6(t)\end{bmatrix}$$

$$\begin{bmatrix}\ddot p_7\\ \ddot p_8\end{bmatrix}+\begin{bmatrix}7 & -3\\ 3 & 8\end{bmatrix}\begin{bmatrix}p_7\\ p_8\end{bmatrix}=\begin{bmatrix}g_7(t)\\ g_8(t)\end{bmatrix}$$

and

$$\ddot p_9 + 9p_9 = g_9(t)$$

where

$$\begin{bmatrix} g_1(t) & g_2(t) & g_3(t) & g_4(t) & g_5(t)\, g_6(t) g_7(t) g_8(t) g_9(t)\end{bmatrix}^T = Q^T f(t)$$

with

$$g_1(t)=\frac{1}{3}\sum_{j=1}^{9} f_j(t),\quad g_2(t)=-\frac{2\sqrt{2}}{3}f_2(t)+\frac{\sqrt{2}}{12}\sum_{j=1, j\neq 2}^{9} f_j(t)$$

$$g_3(t)=-\frac{\sqrt{14}}{4}f_3(t)+\frac{\sqrt{14}}{28}\sum_{\substack{j=1\\ j\neq 2,3}}^{9} f_j(t)$$

$$g_4(t)=-\frac{\sqrt{42}}{7}f_4(t)+\frac{\sqrt{42}}{42}\sum_{\substack{j=1\\ j\neq 2,3,4}}^{9} f_j(t)$$

$$g_5(t)=-\frac{\sqrt{30}}{6}f_5(t)+\frac{\sqrt{30}}{30}\sum_{\substack{j=1\\ j\neq 2,3,4,5}}^{9} f_1(t)$$

$$g_6(t)=-\frac{2\sqrt{5}}{5}f_6(t)+\frac{\sqrt{5}}{10}\sum_{j=1,7,8,9} f_j(t)$$

$$g_7(t)=-\frac{\sqrt{3}}{2}f_7(t)+\frac{\sqrt{3}}{6}\sum_{j=1,8,9} f_j(t)$$

$$g_8(t)=-\frac{\sqrt{6}}{3}f_8(t)+\frac{\sqrt{6}}{6}\sum_{j=1,9} f_j(t), \text{ and}$$

$$g_9(t)=\frac{\sqrt{2}}{2}(f_1(t)-f_9(t))$$

COROLLARY 2. *Let* $K = K^T$, $N = -N^T$, $G = -G^T$ *and* $Rank(G) = 2m \le n$. *If the matrices* K, N *and* G, *commute pairwise, i.e.,*

$$[K, N] = 0, [K, G] = 0, [G, N] = 0 \tag{45}$$

then there exists a real linear change of coordinates that transforms (32) to the form (37) with

$$\Lambda = diag(\lambda_1 I_2, \ldots, \lambda_m I_2, \lambda_{2m+1}, \ldots, \lambda_n) \tag{46}$$

$$\Gamma = diag(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \tag{47}$$

and

$$\mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_{n/2} J_2) \text{ for n even} \tag{48a}$$

$$= diag(\nu_1 J_2, \ldots, \nu_{(n-1)/2} J_2, 0) \text{ for n odd} \tag{48b}$$

where λ_j, ν_j, and β_j *are real numbers and* $\beta_j \neq 0$, $j = 1, \ldots, m$. *If* $m < [n/2]$*-the integer of n/2, then for any* $\nu_{m+k} \neq 0$, $\lambda_{2m+2k-1} = \lambda_{2m+2k}$, $k \in \{1, \ldots, [n/2] - m\}$.

Proof. If the matrices K, N and G commute pairwise, then the conditions in (41)–(43) are satisfied, and, according to Result 1, there exists a real orthogonal transformation Q which transforms (32) to the form given in (37)–(40). Moreover, the first two conditions in (45) correspond to the conditions $[\Lambda, \mathrm{N}] = 0$ and $[\Lambda, \Gamma] = 0$. The condition $[\Lambda, \Gamma] = 0$ requires $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4, \ldots, \lambda_{2m-1} = \lambda_{2m}$ because $\beta_j \neq 0$. If $m < [n/2]$ and $\nu_{m+k} \neq 0$ for some $k \in \{1, \ldots, [n/2] - m\}$, the condition $[\Lambda, \mathrm{N}] = 0$ additionally requires $\lambda_{2m+2k-1} = \lambda_{2m+2k}$. ■

We illustrate this result in the example below.

Example 3. Consider an 11 degrees of freedom system that satisfies the conditions in (45) so that a real orthogonal matrix Q exists that simultaneously quasi-diagonalizes K, G, and N, and the matrices Λ, Γ, and N for the system are given by

$$\Lambda = diag(\lambda_1, \ldots, \lambda_{11}), \Gamma = diag(J_2, J_2, 0_7), \text{and}$$
$$\mathrm{N} = diag(0J_2, 2J_2, 4J_2, 0J_2, J_2, 0)$$

where 0_7 is the $7\, by\, 7$ zero matrix. The non-zero $\beta_j{}'s$ in Γ are $\beta_1 = \beta_2 = 1$, and the non-zero $\nu_j{}'s$ in N are $\nu_2 = 2$, $\nu_3 = 4$, and $\nu_5 = 1$. Since $n = 11$, $[n/2] = 5$. Also, the rank of Γ is 4, so that $m = 2$, and we find that $m < [n/2]$. Then the condition that $[\Lambda, \Gamma] = 0$, requires that $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. Next, from N we see that the values of k for which $\nu_{m+k} \neq 0$ are the set $\{1,3\}$. The condition $[\Lambda, \mathrm{N}] = 0$, then requires that $\lambda_{2m+2k-1} = \lambda_{2m+2k}$ for $k \in \{1, 3\}$. That is, $\lambda_5 = \lambda_6$, and $\lambda_9 = \lambda_{10}$. We note that the commutation conditions impose no restrictions on the remaining $\lambda_j{}'s$, namely, on λ_7, λ_8, and λ_{11}.

Remark 10. The pairwise commutation of K, G, and N given in conditions (45) in Corollary 2 ensures that the conditions in (41)–(43) are all satisfied. However, the reverse is not true, i. e., (41)–(43) does not imply (45). This is because the set of matrices $\{K, G, N\}$ that satisfy (41)–(43) is much "larger" (has higher cardinality) than the set that satisfies (45). As a simple example, when $n = 2$, all 2 by 2 matrices K, G, and N satisfy conditions (41)–(43), while the satisfaction of (45) restricts the matrix K to being proportional to the identity matrix.

According to Lemmas 4–6, the following Result is a direct consequence. It covers situations commonly found in real-life systems that are widely encountered in aerospace, civil, and mechanical engineering.

Result 2. Let $K = K^T$, $N = -N^T$, $G = -G^T$ and $Rank(G) = 2m$. Then (32) can be reduced to (37) with Λ, Γ and N as in (38)–(40) using a real orthogonal transformation

(a) When the eigenvalues of the potential matrix K are distinct, and if and only if

$$[K, GN] = 0, [K, G^2] = 0, [K, GKG] = 0, [K, N^2] = 0, [K, NKN] = 0; \quad (49)$$

(b) When the nonzero eigenvalues of the skew-symmetric matrix G are distinct, and if and only if

$$[G, N] = 0, [K, G^2] = 0, [K, N^2] = 0, [K, NKN] = 0; \quad (50)$$

(c) When the nonzero eigenvalues of G are distinct, the nonzero eigenvalues of N are distinct, and if and only if

$$[G, N] = 0, [K, G^2] = 0, [K, N^2] = 0; \quad (51)$$

(d) When the nonzero eigenvalues of G are distinct, the nonzero eigenvalues of N are distinct, the eigenvalues of K are distinct, and if and only if

$$[K, GN] = 0, [K, G^2] = 0, [K, N^2] = 0 \quad \blacksquare \quad (52)$$

We notice that when the matrices K, G, and N have certain characteristics related to their eigenvalues, the number of necessary and sufficient conditions for their simultaneous orthogonal quasi-diagonalization reduces. Indeed, going back to Example 1, we find that the nonzero eigenvalues of G are distinct, and hence we needed to check only for the four conditions (50).

According to Remark 5 under the *four* conditions (41)–(42), system (32) with $Rank(G) = 2m$ can be reduced by a real orthogonal change of coordinates into m independent two-dimensional subsystems and one $(n - 2m)$-dimensional positional (non-gyroscopic) system. Moreover, if $2m \geq n - 2$, (41)–(42) imply (43), i. e., these conditions are necessary and sufficient for (32) to be orthogonally transformed to the form (37)–(40). In particular, this would be true when G has full rank. Therefore, we have the following result.

Result 3. Let $K = K^T$, $N = -N^T$, $G = -G^T$ and $Rank(G) = 2m \geq n - 2$. Then (32) can be reduced to (37) with Λ, Γ and N as in (38)–(40) using a real orthogonal transformation

(a) If and only if

$$[G, N] = 0, [K, GN] = 0, [K, G^2] = 0, [K, GKG] = 0; \quad (53)$$

(b) When the eigenvalues of the potential matrix K are distinct, and if and only if

$$[K, GN] = 0, [K, G^2] = 0, [K, GKG] = 0; \quad (54)$$

(c) When the nonzero eigenvalues of the skew-symmetric matrix G are distinct, and if and only if

$$[G, N] = 0, [K, G^2] = 0. \quad \blacksquare \quad (55)$$

Since matrices with repeated eigenvalues are non-generic, in many applications to physical systems in aerospace, civil, and mechanical engineering, the eigenvalues of G will be distinct. Hence, when $n - 2m \leq 2$, just the two commutation conditions in (55) would suffice for a real orthogonal matrix Q to exist such that (32) can be reduced to (37).

Example 4. Consider the four-degree-of-freedom system described by (32) in which

$$K = \begin{bmatrix} 2.45 & -0.4 & -0.45 & -0.85 \\ -0.4 & 1.6 & -0.4 & 0 \\ -0.45 & -0.4 & 2.45 & 0.85 \\ -0.85 & 0 & 0.85 & 2.05 \end{bmatrix},$$
$$G = \begin{bmatrix} 0 & -5 & -2 & -4 \\ 5 & 0 & -5 & -2 \\ 2 & 5 & 0 & -4 \\ 4 & 2 & 4 & 0 \end{bmatrix}, N = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{bmatrix} \quad (56)$$

The spectrum of G is $\{\pm 3\sqrt{2}i, \pm 6\sqrt{2}i\}$ and we can apply Result 3-c. We obtain

$$GN = 4\begin{bmatrix} 0 & 2 & -1 & -2 \\ 2 & 1 & 2 & 0 \\ -1 & 2 & 0 & 2 \\ -2 & 0 & 2 & -1 \end{bmatrix} = NG, \text{i.e.}, [G, N] = 0,$$

and

$$KG^2 = 9\begin{bmatrix} -13.6 & -1.6 & 7.2 & 9.2 \\ -1.6 & -8 & -1.6 & 0 \\ 7.2 & -1.6 & -13.6 & -9.2 \\ 9.2 & 0 & -9.2 & -11.6 \end{bmatrix} = G^2K, \text{i.e.}, [K, G^2] = 0.$$

We see that all conditions of Result 3-c are satisfied and the system (32), (56) can be transformed by a real orthogonal transformation into two independent two-dimensional subsystems. Indeed, the orthogonal coordinate transformation $x = Qp$, where

$$Q = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1/\sqrt{2} & 1 & -1/\sqrt{2} \\ -1 & 0 & 0 & -\sqrt{2} \\ 1 & 1/\sqrt{2} & -1 & -1/\sqrt{2} \\ 0 & -\sqrt{2} & -1 & 0 \end{bmatrix}$$

transforms the system into the following uncoupled form that has two independent two-degree-of-freedom subsystems

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + 3\sqrt{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + 2\sqrt{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 2.4p_1 \\ 1.2p_2 \end{bmatrix} = 0$$

$$\begin{bmatrix}\ddot{p}_3\\ \ddot{p}_4\end{bmatrix}+6\sqrt{2}\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\dot{p}_3\\ \dot{p}_4\end{bmatrix}-\sqrt{2}\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}p_3\\ p_4\end{bmatrix}+\begin{bmatrix}3.75p_3\\ 1.2p_4\end{bmatrix}=0.$$

By using (33)–(35), taking into account that the eigenvalues of the matrices G, N and K are same as of $\tilde{M}^{-1}\tilde{G}$, $\tilde{M}^{-1}\tilde{N}$ and $\tilde{M}^{-1}\tilde{K}$ respectively, the Results for the system described by (32) can be translated for (1) as follows.

Result 4. Let $\tilde{M}=\tilde{M}^T>0$, $\tilde{K}=\tilde{K}^T$, $\tilde{N}=-\tilde{N}^T$, $\tilde{G}=-\tilde{G}^T$ and $Rank(\tilde{G})=2m$. The necessary and sufficient conditions that the system described by (1) can be transformed by a linear change of coordinates to the one given in (37)–(40) are that

$$\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{G},\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{G}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K} \quad (57)$$

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K},(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1})^2=(\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2 \quad (58)$$

and

$$\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K},(\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1})^2=(\tilde{N}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2. \quad (59)$$

Result 5. Let $\tilde{M}=\tilde{M}^T>0$, $\tilde{K}=\tilde{K}^T$, $\tilde{N}=-\tilde{N}^T$, $\tilde{G}=-\tilde{G}^T$ and $Rank(\tilde{G})=2m$. Then (1) can be reduced to the (37) with Λ, Γ and N as in (38)–(40) using a real change of coordinates

(a) When the eigenvalues of the matrix $\tilde{M}^{-1}\tilde{K}$ are distinct, and if and only if the following five conditions are satisfied

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{G}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K} \quad (60)$$

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K},(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1})^2=(\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2, \quad (61)$$

$$\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K},(\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1})^2=(\tilde{N}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2 \quad (62)$$

(b) When the nonzero eigenvalues of the matrix $\tilde{M}^{-1}\tilde{G}$ are distinct, and if and only if the following four conditions are satisfied

$$\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{G},\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K} \quad (63)$$

$$\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K},(\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1})^2=(\tilde{N}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2 \quad (64)$$

(c) When the nonzero eigenvalues of $\tilde{M}^{-1}\tilde{G}$ are distinct, the nonzero eigenvalues of $\tilde{M}^{-1}\tilde{N}$ are distinct, and if and only if the following three conditions are satisfied

$$\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{G},\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K},\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K} \quad (65)$$

(d) When the nonzero eigenvalues of $\tilde{M}^{-1}\tilde{G}$ are distinct, the nonzero eigenvalues of $\tilde{M}^{-1}\tilde{N}$ are distinct, the eigenvalues of $\tilde{M}^{-1}\tilde{K}$ are distinct, and if and only if the following three conditions are satisfied

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{G}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K} \quad (66)$$

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K},\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}. \quad (67)$$

Result 6. Let $\tilde{M}=\tilde{M}^T>0$, $\tilde{K}=\tilde{K}^T$, $\tilde{N}=-\tilde{N}^T$, $\tilde{G}=-\tilde{G}^T$ and $Rank(\tilde{G})=2m\geq n-2$. Then (32) can be reduced to (37) with Λ, Γ and N as in (38)–(40) using a real change of coordinates

(a) If and only if

$$\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{G},\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{G}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K} \quad (68)$$

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K},(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1})^2=(\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2; \quad (69)$$

(b) When the eigenvalues of the matrix $\tilde{M}^{-1}\tilde{K}$ are distinct, and if and only if the following three conditions are satisfied

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{G}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K} \quad (70)$$

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K},(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1})^2=(\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2, \quad (71)$$

(c) When the nonzero eigenvalues of the matrix $\tilde{M}^{-1}\tilde{G}$ are distinct, and if and only if the following two conditions are satisfied

$$\tilde{G}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{G},\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K} \quad (72)$$

Example 5. Consider the system (1) with

$$\tilde{M}=\begin{bmatrix}5&0&4\\0&4&0\\4&0&5\end{bmatrix},\quad \tilde{K}=\begin{bmatrix}13&-3&14\\-3&10&3\\14&3&13\end{bmatrix},\quad \tilde{N}=\begin{bmatrix}0&2&0\\-2&0&2\\0&-2&0\end{bmatrix},\tilde{G}=\begin{bmatrix}0&-4&0\\4&0&-4\\0&4&0\end{bmatrix} \quad (73)$$

Observing that any three-dimensional nonzero skew-symmetric matrix has distinct eigenvalues and that in this example $\tilde{N}=-0.5\tilde{G}$, we see that the uncoupling conditions (72) reduce to the single condition

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}.$$

Now we first calculate

$$\tilde{M}^{-1}=\frac{1}{9}\begin{bmatrix}5&0&-4\\0&2.25&0\\-4&0&5\end{bmatrix}$$

and then

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}=4\begin{bmatrix}1&6&-1\\6&-20&-6\\-1&-6&1\end{bmatrix}=(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G})^T=\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}$$

and, according to Result 6-c, there exists a change of coordinates $q=Pp$ that decomposes system (1), (73) into two independent subsystems: one with two degrees of freedom and another with a single degree of freedom. Indeed, the transformation $q = Pp$ with

$$P = \begin{bmatrix} \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{11}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{22}} & -\frac{3}{2\sqrt{11}} & 0 \\ -\frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{11}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$$

reduces system (1), (73) to the form

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + 2\sqrt{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} -2p_1 \\ 3.5p_2 \end{bmatrix} = 0$$

$$\ddot{p}_3 + 3p_3 = 0.$$

4 Uncoupling of Systems Through Imposition of Structure on the Skew-Symmetric Matrix N

Consider now a special form of the circulatory matrix N in (32) given by

$$N = -N^T = \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} a_{rs}K^r G^{2s+1} K^r \tag{74}$$

where a_{rs} are real numbers, and $Rank(G) = 2m$.

The form of this series includes expressions for N like,

$$N = \sum_{s=0}^{m-1} a_s G^{2s+1} \tag{75}$$

and

$$N = \sum_{s=0}^{m-1} b_s K^{2s+1} G^{2s+1} K^{2s+1} \tag{76}$$

as well as simpler sums made up of a few terms, for example

$$N = a_0 G + a_1 KGK + a_3 KG^3 K$$

Result 7. Let $K = K^T$, $N = -N^T$, $G = -G^T$ and $Rank(G) = 2m \leq n$. If and only if

$$[K, G^2] = 0 \tag{77}$$

and

$$[K, GKG] = 0 \tag{78}$$

there exists a real orthogonal coordinate change $x = Qp$ that decomposes (32) with the circulatory matrix of the form given in (74) into m independent, uncoupled two-degree-of-freedom subsystems, and $n-2m$ independent, uncoupled single-degree-of freedom subsystems given by

$$\ddot{p} + \Gamma\dot{p} + \mathrm{N}p + \Lambda p = 0 \tag{79}$$

where

$$\Lambda = diag(\lambda_1, \ldots, \lambda_n) \tag{80}$$

$$\Gamma = diag(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \tag{81}$$

and the circulatory matrix

$$\mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_m J_2, 0_{n-2m}) \tag{82}$$

with

$$\nu_j = \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}(-1)^s a_{rs}\beta_j^{2s+1}\lambda_{2j-1}^r\lambda_{2j}^r, j = 1, \ldots, m \tag{83}$$

Proof. Using the orthogonal transformation $x = Qp$ in (32) and multiplying from the left by Q^T gives

$$\ddot{p} + Q^T GQ\dot{p} + Q^T NQp + Q^T KQp = 0.$$

It follows from Lemma 1 (see Remark 1) that an orthogonal matrix Q exists, such that $Q^T KQ = \Lambda$ and $Q^T GQ = \Gamma$, where Λ and Γ are as in (80) and (81) if and only if conditions (77) and (78) are satisfied. Now, because of (74), it follows that $Q^T NQ = \mathrm{N}$ has form (82)–(83), since

$$\begin{aligned} Q^T K^r G^{2s+1} K^r Q &= \Lambda^r \Gamma^{2s+1} \Lambda^r \\ &= (-1)^s diag(\beta_1^{2s+1}\lambda_1^r\lambda_2^r J_2, \ldots, \\ &\quad \beta_m^{2s+1}\lambda_{2m-1}^r\lambda_{2m}^r J_2, 0_{n-2m}). \end{aligned}$$

■

Corollary 3. *Let $K = K^T$, $N = -N^T$, $G = -G^T$ and $Rank(G) = 2m \leq n$. Suppose that all nonzero eigenvalues of the matrix G are distinct. If and only if*

$$[K, G^2] = 0 \tag{84}$$

there exists an orthogonal coordinate change $x = Qp$ that decomposes (32) with the circulatory matrix of the form given in (74) into m independent, uncoupled two-degree-of-freedom subsystems, and $n-2m$ independent, uncoupled single-degree-of freedom subsystems given by (79)–(83).

Proof. If all nonzero eigenvalues of the matrix G are distinct then, according to Lemma 4, $[K, G^2] = 0$ implies $[K, GKG] = 0$, and the Corollary follows from Result 7. ■

Although the series form given in (74) for the circulatory matrix N is quite versatile since the real coefficients a_{rs} are arbitrary, it does not cover all matrices N that allow quasi-diagonalization of the system described by (32). For example, when $K = \Lambda$ and $G = \Gamma$ as in (80) and (81), respectively, and $N = diag(0_{2m}, \bar{N})$ where $\bar{N}$ is a nonzero $(n-2m)$-dimensional quasi-diagonal skew-symmetric matrix, then the matrix N cannot be expressed in the form (74). Therefore, (74) and the commutation conditions (77) and (78) suffice for a real orthogonal matrix Q to exist so that the real transformation $x = Qp$ decouples the system described by (32) into independent subsystems of at most two degrees of freedom, although when N has the form (74) these commutation conditions are necessary and sufficient for quasi-diagonalization of the system. We will show in the following analysis that if the skew-symmetric matrices G and KGK are simple (i. e., they have distinct eigenvalues)—a situation that commonly arises in applications—then each matrix N that allows quasi-diagonalization of the system is expressible in the form (75) and (76), respectively.

Suppose that all eigenvalues of G are distinct and that there exists a real orthogonal matrix Q such that $Q^T KQ = \Lambda$, $Q^T GQ = \Gamma$, and $Q^T NQ = \mathrm{N}$, where Λ is diagonal,

$$\Gamma = diag(\beta_1 J_2, \ldots, \beta_{n/2} J_2) \text{ when } n \text{ is even} \tag{85a}$$

$$= diag(\beta_1 J_2, \ldots, \beta_{(n-1)/2} J_2, 0) \text{ when } n \text{ is odd}, \tag{85b}$$

and

$$\mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_{n/2} J_2) \text{ when } n \text{ is even} \tag{86a}$$

$$= diag(\nu_1 J_2, \ldots, \nu_{(n-1)/2} J_2, 0) \text{ when } n \text{ is odd}, \tag{86b}$$

in which the $\beta_j{}'s$ are *nonzero* real numbers and $\nu_j{}'s$ could be *arbitrary* real numbers. From Corollary 3, the matrices $K = Q\Lambda Q^T$ and $G = Q\Gamma Q^T$ satisfy (84) (and automatically (78) since all eigenvalues of G are distinct). Now consider the equation

$$\mathrm{N} = \sum_{s=0}^{m-1} a_s \Gamma^{2s+1} \tag{87}$$

where $m = n/2$ and $m = (n-1)/2$ for n even and odd, respectively, and Γ, and N are as in (85) and (86). Upon expansion, taking into account that $J_2^{2s+1} = (-1)^s J_2$, (87) is equivalent to

$$\frac{\nu_i}{\beta_i} = \sum_{s=0}^{m-1} a_s(-\beta_i^2)^s, i = 1, \ldots, m \tag{88}$$

or, in matrix form

$$V_1 a = D_1 c \tag{89}$$

with $a = [a_0, a_1, \ldots, a_{m-1}]^T$, $c = [\nu_1, \nu_2, \ldots, \nu_m]^T$, $D_1 = diag(1/\beta_1, 1/\beta_2, \ldots, 1/\beta_m)$ and V_1 is the Vandermonde matrix of the scalars $-\beta_1^2, -\beta_2^2, \ldots, -\beta_m^2$, i. e.,

$$V_1 = \begin{bmatrix} 1 & -\beta_1^2 & \ldots & (-\beta_1^2)^{m-1} \\ 1 & -\beta_2^2 & \ldots. & (-\beta_2^2)^{m-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -\beta_m^2 & \ldots & (-\beta_m^2)^{m-1} \end{bmatrix}$$

whose determinant is given by

$$\det A = \prod_{1 \le j < k \le m} (\beta_j^2 - \beta_k^2).$$

Because $\beta_k \neq \beta_j$ for $k \neq j$, the matrix V_1 is nonsingular so (89) has a unique solution. Thus, if all eigenvalues of the matrix G are distinct then any skew-symmetric matrix N which allows orthogonal quasi-diagonalization of system (32) can *always* be expressed in the form (75).

The case when all eigenvalues of the skew-symmetric matrix KGK are distinct can be treated similarly.

The equation

$$\mathrm{N} = \sum_{s=0}^{m-1} b_s \Lambda^{2s+1} \Gamma^{2s+1} \Lambda^{2s+1} \tag{90}$$

where Λ, Γ, and N are same as above, is equivalent to the equation

$$V_2 b = D_2 c \tag{91}$$

with $b = [b_0, b_1, \ldots, b_{m-1}]^T$, $c = [\nu_1, \nu_2, \ldots, \nu_m]^T$, $D_2 = diag(1/\sigma_1, 1/\sigma_2, \ldots, 1/\sigma_m)$, $\sigma_j = \beta_j \lambda_{2j-1} \lambda_{2j} \neq 0$, $j = 1, \ldots, m$, and V_2 is the Vandermonde matrix of the numbers $-\sigma_1^2, -\sigma_2^2, \ldots, -\sigma_m^2$, which is nonsingular since $\sigma_k \neq \sigma_j$ for $k \neq j$. Note the fact that all the numbers σ_j are nonzero. That they are distinct, follows from the assumption that the matrix KGK is simple. Therefore the matrix V_2 is nonsingular and equation. (91) has a unique solution. We therefore conclude that if the eigenvalues of the matrix KGK are distinct then any skew-symmetric matrix N that allows orthogonal quasi-diagonalization of system (32) can *always* be expressed in the form (76).

Now we have the following two results.

Result 8. When all the eigenvalues of the skew-symmetric matrix G are distinct, then the conditions given by (75) and (84) are necessary and sufficient for quasi-diagonalization of (32) using an orthogonal coordinate transformation.

Result 9. When all the eigenvalues of the skew-symmetric matrix KGK are distinct, then the conditions given by (76)–(78) are necessary and sufficient for quasi-diagonalization of (32) using an orthogonal coordinate transformation.

Let us go back to Example 4. In that example, all the eigenvalues of the skew-symmetric matrix G are distinct, and so the system can be quasi-diagonalized by an orthogonal transformation as we have shown earlier. In addition, from Result 8 the skew-symmetric matrix N can be expressed in the form (75). More precisely, it is easy to verify that

$$N = \frac{17}{18} G + \frac{5}{324} G^3.$$

Remark 11. The roles of G and N can be interchanged in the above results starting with (74).

We return to Example 1 to illustrate this Remark. For the matrices K and N from (44) we have

$$KNK = \frac{1}{8} \begin{bmatrix} 0 & 0 & 7 & 1 \\ 0 & 0 & 1 & 7 \\ -7 & -1 & 0 & 0 \\ -1 & -7 & 0 & 0 \end{bmatrix}.$$

Now it is easy to see that the matrix G from (44) can be expressed as

$$G = 16N - 16KNK$$

which is of the form (74) in which the roles of G and N have been interchanged. Since $[K, N^2] = 0$ and $[K, NKN] = 0$, as we have shown earlier, then because of Result 7 in which the matrices G and N are interchanged, the system of this example can be orthogonally quasi-diagonalized. On the other hand, $\sigma(KNK) = \{\pm i, \pm 0.75i\}$, i.e., the matrix KNK has all distinct eigenvalues, and according to Result 9 (with G and N interchanged) the matrix G can be expressed in the form

$$G = b_1 KNK + b_2 K^3 N^3 K^3.$$

We find that $b_1 = b_2 = 256/21$.

The sums (74)–(76) can be expressed in terms of the matrices $\tilde{M}$, $\tilde{K}$, $\tilde{G}$ and $\tilde{N}$ of the original dynamical system (1) using (33)–(35), and all the above statements obtained for the system described by (32) can be translated for (1). For example, (75) becomes

$$\tilde{N} = \sum_{j=1}^{m} a_j (\tilde{G}\tilde{M}^{-1})^{2(j-1)} \tilde{G} \tag{92}$$

and the following result can be formulated.

Result 10. When the matrix $\tilde{M}^{-1}\tilde{G}$ has all distinct eigenvalues, then the condition

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K} \tag{93}$$

together with (92), is necessary and sufficient for quasi-diagonalization of (1) using a real change of coordinates.

Example 6. Consider a three-degree-of-freedom system (1) with nonzero matrix $\tilde{G}$. Since the matrix $\tilde{M}^{-1}\tilde{G}$ has distinct eigenvalues and $m = 1$, it follows, according to Result 10, that the system can be quasi-diagonalized by a real change of coordinates if and only if $\tilde{N} = a_1 \tilde{G}$, where a_1 is a real number, and condition (93) is satisfied.

5 Conclusions

This paper develops a new central result in linear algebra that identifies the necessary and sufficient (n&s) conditions for the simultaneous quasi-diagonalization of two skew-symmetric matrices and a symmetric matrix using an orthogonal congruence. This result is used to quasi-diagonalize a linear gyroscopic nonconservative n-degree-of-freedom (MDOF) system described by a mass matrix that is normalized to the identity matrix I, a gyroscopic skew-symmetric matrix (G), a circulatory skew-symmetric matrix (N), a potential symmetric matrix (K), with G multiplying the velocities to give the gyroscopic force. Since any arbitrary stiffness matrix, S, can be split into its additive symmetric and skew-symmetric parts as $S = K + N$, the paper addresses the uncoupling of gyroscopic systems with arbitrary (nonconservative) stiffness matrices. Both forced and free vibrations of the MDOF system are considered.

It is shown that a total of six n&s commutation conditions between the three n by n matrices K, G, and N are needed for their simultaneous orthogonal quasi-diagonalization. This simultaneous orthogonal quasi-diagonalization, which is achieved by a real coordinate transformation through the use of a real orthogonal matrix, yields a set of uncoupled, independent subsystems in which each subsystem has no more than two degrees of freedom. These n&s commutation conditions, of course, place restrictions on the three matrices. When K has distinct eigenvalues, the number of n&s commutation conditions reduce to 5; when the non-zero eigenvalues of $G(N)$ are distinct, they reduce to 4; and, when the non-zero eigenvalues of both G and N are each distinct, they reduce to 3. Recalling that it is generic to have distinct eigenvalues, many of these conditions are often met in real-life structural and mechanical systems found in aerospace, civil, and mechanical engineering. Because the behavior of a two-degree-of-freedom subsystem is much simpler to understand than that of an MDOF system with numerous degrees of freedom, the decomposition of an MDOF system into such subsystems is useful in providing a better understanding of the MDOF system's behavior as well as in providing more accurate computational methods in the determination of its response to external forces.

Furthermore, when the rank, $2m$, of the skew-symmetric matrix $G(N)$ is such that $n - 2m \leq 2$, where n is the number of degrees of freedom of the MDOF system, then the number of n&s conditions, for uncoupling the MDOF system into a set of independent subsystems, each with at most two degrees-of-freedom by an orthogonal coordinate change, reduces to four. This reduces to three n&s conditions when the eigenvalues of K are distinct, and then further down to just two n&s conditions when the non-zero eigenvalues of $G(N)$ are distinct.

In an effort to reduce the number of n&s conditions for simultaneous orthogonal quasi-diagonalization, we posit a specific form for the matrix $N(G)$ and show that the number of n&s commutation conditions required reduces to just two. When $N(G)$ has this posited form and the nonzero-eigenvalues of $G(N)$ are distinct a *single* n&s condition permits the MDOF system to be decomposed into a set of at most two degree-of-freedom independent subsystems. It is shown that when all the eigenvalues either of the gyroscopic matrix $G(N)$ or of the matrix $KGK(KNK)$ are distinct—a situation, as mentioned before, that often occurs in real-life—, all gyroscopic nonconservative multi-degree-of-freedom systems that can be orthogonally quasi-diagonalized to yield uncoupled subsystems of dimension two or less must have this posited form for their circulatory matrix $N(G)$.

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

No data, models, or code were generated or used for this paper.

Appendix

LEMMA 3. *The four commutation conditions given in (15)–(17), which are*

$$[G, N] = 0, [K, GN] = 0, [K, G^2] = 0, [K, GKG] = 0 \qquad \text{(A1)}$$

guarantee the remainder of the three pairwise commutation conditions among the matrices

$$K, GN, G^2, \text{and } GKG$$

given in (19).

Proof. We begin by noting that for any three n by n matrices, A, B, and C, the following properties are true:

(a) $[A, B] = -[B, A]$, and $[A, B] = 0$ implies that $[B, A] = 0$,
(b) $[AB, C] = [A, C]B + A[B, C]$ *and* $[A, BC] = [A, B]C + B[A, C]$
(c) $[A^2, A] = 0$

Using (1) and properties (a)–(c), we next obtain:

$$[GN, G] = G[N, G] + [G, G]N = -G[G, N] + [G, G] = 0 \qquad \text{(A2)}$$

$$[GN, KG] = [GN, K]G + K[GN, G] = -[K, GN]G + K[GN, G] = 0 \qquad \text{(A3)}$$

We note that (A2) requires that only the first commutation condition in (A1) be satisfied and (A3) requires that only the first two commutation conditions in (A1) be satisfied.

We now show the following three remaining commutators equal to zero when (A1) is true. We will be using property (b) often.

(i) $[GN, G^2] = [GN, G]G + G[GN, G] = 0$

In the last equality, we have used (A2).

(ii) $[GN, GKG] = [GN, G]KG + G[GN, KG] = 0$
To get the last equality we use (A2) and (A3).

(iii) $$\begin{aligned}[G^2, GKG] &= [G^2, G]KG + G[G^2, KG] \\ &= G[G^2, KG] = G[G^2, K]G + GK[G^2, G] \\ &= G[G^2, K]G = -G[K, G^2]G = 0.\end{aligned}$$

We have used property (c) to get the second and fourth equalities.

We observe that (i) follows from the first commutation condition in (A1), (ii) follows from the first two commutation conditions, and (iii) follows from the third commutation condition in (A1). ∎

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